

# Local electromagnetic duality and gauge invariance

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Bunster and Henneaux and, separately, Deser have very recently considered the possibility of gauging the usual electromagnetic duality of Maxwell equations. By using off-shell manipulations in the context of the Principle of least action, they conclude that this is not possible, at least with the conventional compensating gauge fields scheme. Such a conclusion contradicts, however, an old result of Malik and Pradhan, who showed that it is indeed possible to introduce an extra abelian gauge field directly in the vacuum Maxwell equations in order to render them covariant under local duality transformations. Since it is well known that the equations of motion can, in general, admit more symmetries than the associate Lagrangian, this would not be a paradoxal result, in principle. Here, we revisit these works and identify the source of the different conclusions. We show that the Malik-Pradhan procedure does not preserve the original Maxwell gauge invariance, while Bunster, Henneaux, and Deser sought for generalizations which are, by construction, invariant under the Maxwell gauge transformation. Further, we show that the Malik-Pradhan construction cannot be adapted or extended in order to preserve the Maxwell gauge invariance, reinforcing the conclusion that it is not possible to gauge the electromagnetic duality.

PACS numbers: 03.50.De, 11.15.-q

## I. INTRODUCTION

It is well known that Maxwell equations in vacuum (natural units are adopted through this work)

$$\begin{aligned}\nabla \cdot \mathbf{E} &= 0, \quad \frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} = 0, \\ \nabla \cdot \mathbf{B} &= 0, \quad \frac{\partial}{\partial t} \mathbf{E} - \nabla \times \mathbf{B} = 0,\end{aligned}\tag{1}$$

are invariant under the electromagnetic duality transformation [1, 2]

$$\begin{aligned}\mathbf{E} &\rightarrow \mathbf{E}' = \cos \theta \mathbf{E} - \sin \theta \mathbf{B}, \\ \mathbf{B} &\rightarrow \mathbf{B}' = \cos \theta \mathbf{B} + \sin \theta \mathbf{E}.\end{aligned}\tag{2}$$

The study of the electromagnetic duality has led to numerous advances in quantum mechanics and field theory, ranging from the Dirac charge quantization to the introduction of S-duality in string-inspired models, see, for instance, [2, 3]. Very recently, Bunster and Henneaux [4] and, separately, Deser [5] investigated the problem of gauging the electromagnetic duality, *i.e.*, to consider that, instead of the rigid rotation (2), the angle  $\theta$  could depend on the spacetime position. By using off-shell manipulations in the electromagnetic action as done, for instance, in [6, 7], they conclude that the electromagnetic duality cannot be gauged, at least by means of the conventional compensating gauge fields scheme.

Nevertheless, exactly the same question about the gauging of the duality (2) was raised 25 years ago by Malik and Pradhan [8]. Surprisingly, they show that the Maxwell equations (1) can indeed be modified by introducing a compensation massless vector field in order to

accommodate a local duality transformation. Their analysis was based on the equations of motion, and not on the Principle of least action as the more recent works [4, 5] were. Since the equations of motion can have in general more symmetries than the corresponding Lagrangian [9–11], such a result would not be, in principle, a paradox. Let us briefly recall the Malik-Pradhan construction in order to clarify why they got a distinct conclusion. For our purposes here, it is more convenient to introduce the complex electromagnetic vector

$$\mathbf{F} = \mathbf{E} + i\mathbf{B},\tag{3}$$

which, incidentally, can be traced back to the XIX century, see[2, 12]. Maxwell equations in vacuum for the complex electromagnetic vector (3) read simply as

$$\nabla \cdot \mathbf{F} = 0, \quad i \frac{\partial}{\partial t} \mathbf{F} - \nabla \times \mathbf{F} = 0,\tag{4}$$

and the local duality transformation will be given by

$$\mathbf{F} \rightarrow \mathbf{F}' = e^{i\theta(x)} \mathbf{F}.\tag{5}$$

It is clear that (4) is not invariant nor covariant under the local duality transformation (5) since

$$\begin{aligned}\partial_t \mathbf{F} \rightarrow \partial_t \mathbf{F}' &= e^{i\theta(x)} (\partial_t \mathbf{F} + i\partial_t \theta \mathbf{F}), \\ \nabla \cdot \mathbf{F} \rightarrow \nabla \cdot \mathbf{F}' &= e^{i\theta(x)} (\nabla \cdot \mathbf{F} + i\nabla \theta \cdot \mathbf{F}), \\ \nabla \times \mathbf{F} \rightarrow \nabla \times \mathbf{F}' &= e^{i\theta(x)} (\nabla \times \mathbf{F} + i\nabla \theta \times \mathbf{F}).\end{aligned}\tag{6}$$

However, this obstruction can be easily circumvented by introducing a real four vector  $a_\mu = (a_0, \mathbf{a})$  transforming as

$$a_\mu \rightarrow a'_\mu = a_\mu + \partial_\mu \theta\tag{7}$$

under (5). The operators  $\mathcal{D}_t = \partial_t - ia_0$  and  $\mathcal{D} = \nabla - i\mathbf{a}$  transform under (5) and (7) in the correct way

$$\mathcal{D}_t \mathbf{F} \rightarrow \mathcal{D}'_t \mathbf{F}' = e^{i\theta(x)} \mathcal{D}_t \mathbf{F},$$

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$$\begin{aligned}\mathcal{D} \cdot \mathbf{F} \rightarrow \mathcal{D}' \cdot \mathbf{F}' &= e^{i\theta(x)} \mathcal{D} \cdot \mathbf{F}, \\ \mathcal{D} \times \mathbf{F} \rightarrow \mathcal{D}' \times \mathbf{F}' &= e^{i\theta(x)} \mathcal{D} \times \mathbf{F},\end{aligned}\quad (8)$$

rendering the equations

$$(\nabla - i\mathbf{a}) \cdot \mathbf{F} = 0, \quad (9)$$

$$i \left( \frac{\partial}{\partial t} - ia_0 \right) \mathbf{F} - (\nabla - i\mathbf{a}) \times \mathbf{F} = 0, \quad (10)$$

covariant under the transformations (5) and (7). Eqs. (9) and (10) have interesting properties which we shall discuss later, but they have a serious drawback lying at the heart of the contradiction between the works [4, 5] and [8]: the usual Maxwell gauge invariance is irremediably lost.

## II. TWO GAUGE INVARIANCES

The covariance of Malik-Pradhan (MP) generalized Maxwell equations (9) and (10) under the local duality transformation (5) and (7) can be summarized as follows: if  $(a_\mu, \mathbf{F})$  is a valid solution of the MP equations, then  $(a_\mu + \partial_\mu \theta, e^{i\theta} \mathbf{F})$  is another one. The four-vector field  $a_\mu$  behaves as an abelian gauge field and, hence, any quantity based on the tensor

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu \quad (11)$$

is invariant under the local duality gauge invariance. The dynamics for the gauge field  $a_\mu$  can be introduced by exploring the invariant tensor (11). There are two evident electromagnetic invariants under the local duality transformation, namely the electromagnetic energy density and the Poynting vector

$$\begin{aligned}\frac{1}{2} \mathbf{F}^* \cdot \mathbf{F} &= \frac{1}{2} (|\mathbf{E}|^2 + |\mathbf{B}|^2), \\ \frac{1}{2i} \mathbf{F}^* \times \mathbf{F} &= \mathbf{E} \times \mathbf{B},\end{aligned}\quad (12)$$

where  $\mathbf{F}^* = \mathbf{E} - i\mathbf{B}$ . Several other interesting properties of (9) and (10) are discussed in [8].

The MP equations (9) and (10) are a legitimate generalization of Maxwell equation with respect to the local electromagnetic duality. Moreover, they can be considered as the simplest possible generalization, obtained by means of the usual minimal coupling procedure, with the introduction of the covariant derivative  $\partial_\mu \rightarrow \partial_\mu - ia_\mu$ . However, they lost their distinctive Maxwell gauge invariance. A solution of the usual Maxwell equations corresponds to a  $(0, \mathbf{F})$  solution of the MP equations. For  $a_\mu = 0$ , Eq. (9) implies that

$$\mathbf{F} = \nabla \times \mathbf{A}, \quad (13)$$

where  $\mathbf{A}$  is a complex vector potential with real and complex parts corresponding, respectively, to the electric and

magnetic vector potentials. The electromagnetic vector defined by (13) is invariant under the usual Maxwell gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\phi, \quad (14)$$

for a complex function  $\phi$ . In the variational formulation considered in [4, 5], the first complex Maxwell equation in (4) and its solution (13) are taken for granted, and the second complex Maxwell equation is then obtained from the minimization of the electromagnetic action with respect to the variable  $\mathbf{A}$ , assuring by construction, in this way, the invariance of the whole system under (14). It is clear now the origin of the distinct conclusions: while Bunster and Henneaux [4], and Deser [5] have been looking for extensions of the gauge-invariant electromagnetic action, Malik and Pradhan [8] modified all the Maxwell equations, but spoiled the Maxwell gauge invariance.

The Malik-Pradhan construction is, indeed, incompatible with the Maxwell gauge invariance (14). First, Eq. (9) does not admit (13) as a solution in general and, consequently, the transformation (14) cannot be a symmetry of the system. One could try to modify the MP construction by adopting a procedure similar to that one used in the Principle of least action, attempting to keep the Maxwell gauge invariance by the construction. Let us assume the form (13) and look for a generalization of the second complex Maxwell equation. If the electromagnetic vector transform as  $\mathbf{F} \rightarrow e^{i\theta} \mathbf{F}$  under the duality transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = e^{i\theta(x)} \mathbf{A}, \quad (15)$$

Eq. (10) will be the right generalization of the second Maxwell equation. This can be achieved by generalizing (13) to

$$\mathbf{F} = (\nabla - i\mathbf{a}) \times \mathbf{A}, \quad (16)$$

and taking into account (7). However, (16) is not invariant under (14) anymore and, moreover, it cannot be cast in covariant form by introducing gauge compensation fields guided by the minimal coupling procedure. Under (14), we have  $\mathbf{F} \rightarrow \mathbf{F} - i\mathbf{a} \times \nabla\phi$  from (16). Such an extra term could be absorbed with the introduction of a gauge scalar field, but in this case the local duality invariance would be spoiled. In fact, we can show that no local theory of this type can be simultaneously invariant under the gauge transformations  $\mathbf{A} \rightarrow \mathbf{A} + \nabla\phi$  and covariant under  $\mathbf{A} \rightarrow e^{i\theta} \mathbf{A}$ . Suppose we have two equivalent potentials with respect to (14), say  $\mathbf{A}$  and  $\mathbf{A} + \nabla\phi$ . Now, applying for these two equivalent potentials the gauge transformation (15) we end up, respectively, with  $e^{i\theta} \mathbf{A}$  and  $e^{i\theta} \mathbf{A} + e^{i\theta} \nabla\phi$ , which do not belong anymore to the same equivalent class since  $e^{i\theta} \nabla\phi \neq \nabla\phi'$  in general. Notice that, if we give up the generalization (16), we will have no transformation  $\mathbf{A} \rightarrow \mathbf{A}'$  implying  $\mathbf{F} \rightarrow e^{i\theta} \mathbf{F}$  since it would suppose that

$$\nabla \times \mathbf{A}' = e^{i\theta(x)} \nabla \times \mathbf{A}, \quad (17)$$

which has no solution for general  $\theta(x)$  either. It is not possible to keep the Maxwell gauge invariance along the Malik-Pradhan construction, reinforcing Bunster, Henneaux, and Deser recent conclusion that it is not possible to gauge the electromagnetic duality.

It is interesting to notice that, giving up of a local theory, it is indeed possible [13] to construct a formal solution of (9) simultaneously covariant under (5) and invariant under (14)

$$\mathbf{F}_{\Gamma_x} = \exp \left( i \int_{\Gamma_x} a_\mu dx^\mu \right) \nabla \times \mathbf{A}, \quad (18)$$

with the assumption that  $\mathbf{A}$  is invariant under the local duality, where  $\Gamma_x$  is a continuous world line ending at the spacetime point  $x = (t, \mathbf{x})$ . However, for general  $a_\mu$ , the electromagnetic vector (18) does depend on the integration path  $\Gamma_x$ . Such non-locality implies severe non-uniqueness, challenging the physical applicability of non-local extensions of this type. On the other hand, for the case of pure gauge field  $a_\mu = \partial_\mu \theta$ , Eq. (18) defines a unique electromagnetic vector  $\mathbf{F} = e^{i\theta} \nabla \times \mathbf{A}$ .

### III. FINAL REMARKS

Even though the serious drawback of the breaking of Maxwell gauge invariance, the Malik-Pradhan equations (9) and (10) have some interesting properties. In terms of the original electric and magnetic fields, they read

$$\begin{aligned} \nabla \cdot \mathbf{E} &= -\mathbf{a} \cdot \mathbf{B}, \quad \nabla \cdot \mathbf{B} = \mathbf{a} \cdot \mathbf{E} \\ \frac{\partial}{\partial t} \mathbf{B} + \nabla \times \mathbf{E} &= a_0 \mathbf{E} - \mathbf{a} \times \mathbf{B}, \\ \frac{\partial}{\partial t} \mathbf{E} - \nabla \times \mathbf{B} &= -a_0 \mathbf{B} + \mathbf{a} \times \mathbf{E}. \end{aligned} \quad (19)$$

Without loss of generality, one can set  $a_0 = 0$  by exploring the local duality transformations. Malik and Pradhan suggested, in this case, a connection between (19) and the Maxwell equations in a magneto-electric medium, *i.e.*, a material medium with a linear and reciprocal relationship between the magnetic field and the electric polarization, and between the electric field and the magnetic polarization, besides the usual relationships between the magnetic field and the magnetic polarization and between the electric field and electric polarization[14]

$$\begin{aligned} \mathbf{H} &= \mu^{-1} \mathbf{B} - \beta(\mathbf{x}) \mathbf{E}, \\ \mathbf{D} &= \epsilon \mathbf{E} + \beta(\mathbf{x}) \mathbf{B}. \end{aligned} \quad (20)$$

However, Eq. (23) in [8] requires, necessarily, the existence of magnetic charges and currents, compromising the physical interpretation of (19) in terms of realistic electrodynamics in material media. In order to save such an interpretation, we can restrict our analysis to solutions such that  $\mathbf{E}$  and  $\mathbf{B}$  are orthogonal and  $\mathbf{a}$  is parallel to  $\mathbf{B}$ . Compare now this particular case of (19) with the

Maxwell equations in a medium obeying (20)

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \\ \epsilon \nabla \cdot \mathbf{E} &= -\nabla \beta \cdot \mathbf{B}, \\ \frac{\partial}{\partial t} \mathbf{E} - \nabla \times \mathbf{B} &= 0, \\ \mu^{-1} \frac{\partial}{\partial t} \mathbf{B} + \epsilon \nabla \times \mathbf{E} &= -\nabla \beta \times \mathbf{B}. \end{aligned} \quad (21)$$

The MP equations for such class of solutions are identical to the Maxwell equations in a magneto-electric medium such that  $\mu^{-1} = \epsilon$  and  $\mathbf{a} = \epsilon^{-1} \nabla \beta$ . Notice that, as in the example considered by Malik and Pradhan, the vector  $\mathbf{a}$  is a pure gauge field in this case, suggesting that the space dependent mixing of the electric and magnetic fields in (20) is, in fact, nothing more than a gauge effect! This point certainly deserves a deeper investigation, mainly due to the experimental relevance of magneto-electric media, see [15] for a recente review.

Yet for the case of a pure gauge field  $a_\mu = \partial_\mu \theta$ , we have another interesting geometrical interpretation for the MP construction. The covariant derivatives with respect to the local duality can be rewritten, in this case, by using the differential operator

$$\mathcal{D}_\mu = \partial_\mu - i \partial_\mu \theta = e^{i\theta} \partial_\mu e^{-i\theta}. \quad (22)$$

Such kind of covariant derivatives are related to deformations of the spacetime volume element and they have already been used to study the compatibility between the minimal coupling procedure and the Principle of least action for gauge fields in Riemann-Cartan spacetimes [16]. In particular, the (complex) scalar field  $\theta$  has a similar dynamics to the dilaton field[17]. Notice that another particular case of the MP construction involving a pure gauge field  $a_\mu = \partial_\mu \theta$  was independently proposed by Kato and Singleton in [18], where they also recognize the incompatibility between invariance under the usual gauge transformations  $\mathbf{A} \rightarrow \mathbf{A} + \nabla \phi$  and covariance under the local duality transformation  $\mathbf{A} \rightarrow e^{i\theta} \mathbf{A}$ .

We finish this note with two remarks. First, it is clear that the Malik-Pradhan construction is intrinsically four (spacetime) dimensional due to the use of many three (space) dimensional identities in its derivation. Indeed, one does not expect any electromagnetical duality transformation for other dimensions, only in four spacetime dimensions one can have a duality between the magnetic and electric three-vectors. Second, in contrast to the original Maxwell equations, in the Malik-Pradhan equations the symmetry between self-dual ( $\mathbf{F}^* = 0$ ) and anti-self-dual ( $\mathbf{F} = 0$ ) solutions is broken, *i.e.*, if  $(a_\mu, \mathbf{F})$  is a self-dual solution of (9) and (10), the corresponding anti-self-dual solution will be  $(-a_\mu, \mathbf{F}^*)$ . They are equivalent up to a local duality transformation only if  $a_\mu$  is a pure gauge, *i.e.*, only if  $f_{\mu\nu} = 0$ .

## Acknowledgments

This work was supported by FAPESP and CNPq. It is a pleasure to thank E. Abdalla, S. Deser, and D. Sin-

gleton for enlightening discussions and useful communications.

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